

Winding angle distribution of 2D random walks with traps

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We study analytically the asymptotic behaviour of the average probability $\mathcal{P}(n, t)$ for the trajectory of a 2D Brownian particle wandering in the presence of randomly distributed traps to wind n times around a given point after a time t . It is shown that $\mathcal{P}(n, t) \sim \exp(-c\sqrt{t})(1+x^2)^{-1}$ with $x \sim n/\sqrt{t}$, where the first exponent represents a well-known long-time tail of the probability that a particle will not be trapped.

The properties of random walks with various topological constraints has attracted a great deal of theoretical interest for many years. Apart from apparent practical relevance to the physics of polymers or Abrikosov vortex lines in superconductors (see, e.g., Ref. [1]), studies of such random walks make profound and intimate connections to many beautiful mathematical results. Perhaps, the most prominent example is the problem of the winding angle distribution in two dimensions. The winding angle of a planar random walk is, by definition, the total continuous angle $\theta(t) = 2\pi n(t)$ swept by a Brownian particle around a prescribed point after a time t . It was found by Spitzer [2] that the asymptotic probability to wind n times is given by a Cauchy law:

$$\mathcal{P}(n, t) \sim \frac{1}{1+x^2}, \quad x \sim \frac{n}{\ln t}, \quad \text{at } t \rightarrow \infty. \quad (1)$$

This result was later confirmed by many authors by employing different techniques (see, e.g., Refs. [3–5]). In general, one could also ask how many times a particle has wound around a set of N prescribed points [6] or an excluded disk in a 2D plane [7].

Let us now suppose that our Brownian particle can not wander freely, but instead can be irreversibly trapped by the impurities located at some randomly distributed points in a plane. It is known that the properties of such a system differ drastically from those of an ideal random walk. For instance, the survival probability is given by $P(t) \sim \exp(-c\sqrt{t})$ [8], while the mean square displacement is sub-diffusional: $\langle r^2 \rangle \sim \sqrt{t}$ [9]. The purpose of the present article is to discover how the presence of traps affects the winding angle distribution function.

The probability distribution for a random walk starting at a point \mathbf{r}' to end at a point \mathbf{r} after a time t satisfies the diffusion equation

$$\frac{\partial P}{\partial t} = D\nabla^2 P - U(\mathbf{r})P. \quad (2)$$

Here D is the diffusion coefficient and $U(\mathbf{r}) = U_0 \sum_i \delta(\mathbf{r} - \mathbf{R}_i)$ is the random “potential”, which is the probability per unit time for a particle to be trapped ($U_0 > 0$). The positions \mathbf{R}_i of point-like traps are distributed uniformly in a plane according to the Poisson law with mean density ρ . The solution of Eq. (2) is given by the Wiener path integral formula:

$$P(\mathbf{r}, t; \mathbf{r}', 0|U) = \int_{\mathbf{r}(0)=\mathbf{r}'}^{\mathbf{r}(t)=\mathbf{r}} \mathcal{D}\mathbf{r}(\tau) \exp \left\{ - \int_0^t d\tau \left(\frac{1}{4D} \dot{\mathbf{r}}^2(\tau) + U(\mathbf{r}(\tau)) \right) \right\}. \quad (3)$$

The probability for a closed random walk with $\mathbf{r} = \mathbf{r}'$ of “length” t to wind n times around the origin in a given distribution of traps can be calculated by inserting a δ -function constraint [3] in (3), so that

$$\mathcal{P}(n, t|U) = \left\langle \delta \left(n - \frac{1}{2\pi} \int_0^t d\tau \dot{\theta}(\tau) \right) \right\rangle_{P(\mathbf{r}, t; \mathbf{r}, 0|U)}, \quad (4)$$

where $\theta(t)$ is the angle between the radius-vector $\mathbf{r}(t)$ and some fixed direction in the plane (note that n can be non-integer). Writing the δ -function as an integral over an auxiliary variable p , we arrive at

$$\mathcal{P}(n, t|U) = \int_{-\infty}^{\infty} dp e^{2\pi i p n} \int_{\mathbf{r}(0)=\mathbf{r}}^{\mathbf{r}(t)=\mathbf{r}} \mathcal{D}\mathbf{r}(\tau) \exp \left\{ - \int_0^t d\tau \left(\frac{1}{4D} \dot{\mathbf{r}}^2(\tau) + U(\mathbf{r}(\tau)) + i p \dot{\theta}(\tau) \right) \right\}. \quad (5)$$

If one assumes that the starting point \mathbf{r} is not fixed, then the path integral on the right-hand side is nothing but the partition function at inverse “temperature” $1/T = t$ of a particle of unit charge and mass $m = (2D)^{-1}$ moving

in a random potential and in a solenoid field localised at the origin, the solenoid carrying a flux $\phi = -2\pi p$. For the average probability we then have

$$\mathcal{P}(n, t) \equiv \langle \mathcal{P}(n, t|U) \rangle_U = \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \int_0^{\infty} dE e^{-i\phi n} e^{-Et} N(E, \phi), \quad (6)$$

where $N(E, \phi)$ is the average density of states. Since we are interested in the asymptotic behaviour of $\mathcal{P}(n, t)$ at large t , all we have to do is to calculate the asymptotics of $N(E, \phi)$ at small E , which is called ‘‘Lifshitz tail’’ [10].

From the analysis of a random walk with traps but without solenoid it is known that at $E \rightarrow 0$ the main contribution to the density of states comes from the large regions in real space which are free of traps. The probability to find such a region of area S is exponentially small: $p(S) \sim e^{-\rho S}$. From the elementary quantum mechanics we know that the ground state energy of a particle in a 2D potential well with radius R is given by $E(R) \sim DR^{-2} \sim DS^{-1}$. Therefore, $S(E) \sim D/E$, and the density of states is $N(E) \sim p(S(E)) \sim \exp(-\text{const } \rho D/E)$. More formally, such exponentially small tails of the density of states correspond to the contribution of instantons [11], which are spatially localised solutions of the saddle-point equations in the functional-integral representation of the problem [12,13]. Our strategy is as follows. First, we formulate the problem in the language of quantum field theory with some effective action S . Then, we find an explicit form of the instanton solution. The last step is to calculate the asymptotic behaviour of $\mathcal{P}(n, t)$ at large t due to the instanton contributions.

The Schrödinger equation for a quantum particle moving in the field of a solenoid and in the potential $U(\mathbf{r})$ is as follows:

$$H\psi \equiv D(-i\nabla - \mathbf{A}(\mathbf{r}))^2\psi + U(\mathbf{r})\psi = E\psi, \quad (7)$$

where $A_\theta = \phi/2\pi r$ is the vector potential created by the solenoid. The density of states is proportional to the imaginary part of the Green function $G_E(\mathbf{r}, \mathbf{r}) = \langle \mathbf{r} | (E - H + i0)^{-1} | \mathbf{r} \rangle$, which can be calculated by standard means of the quantum field theory. Using the replica trick, the disorder average of the (Euclidean) generating functional can be performed, and we have in the limit $n \rightarrow 0$

$$G_E(\mathbf{r}, \mathbf{r}) = \int \mathcal{D}^2\boldsymbol{\varphi}(\mathbf{r}) e^{-S[\boldsymbol{\varphi}(\mathbf{r})]} \varphi_1(\mathbf{r}) \bar{\varphi}_1(\mathbf{r}), \quad (8)$$

where $\boldsymbol{\varphi}$ is an n -component Bose field and $\mathcal{D}^2\boldsymbol{\varphi} = \prod_{a=1}^n \mathcal{D}\bar{\varphi}_a \mathcal{D}\varphi_a$. The action is

$$S = \int d^2r \left\{ \bar{\varphi} \left(-E + D(-i\nabla - \mathbf{A})^2 \right) \varphi + \rho \left(1 - e^{-U_0 \bar{\varphi} \varphi} \right) \right\}. \quad (9)$$

At $E < 0$ the action is always positive, so that the field theory is stable and the imaginary part is zero. At $0 < E \ll E_c = \rho U_0$ (E_c being the mean value of the random potential) there is a metastable vacuum state $\boldsymbol{\varphi} = 0$. In this case, the small- E asymptotics of the imaginary part of the Green function is determined by a non-trivial saddle point of the action [11] and, with exponential accuracy,

$$N(E, \phi) \sim e^{-S_{\text{inst}}(E, \phi)}. \quad (10)$$

Due to the rotational symmetry of Eq. (9) in the n -dimensional replica space, the saddle-point solution (instanton) has the form

$$\varphi_a(\mathbf{r}) = \varphi(\mathbf{r}) e_a, \quad (11)$$

where e_a is the a -th component of an arbitrary n -component unit vector. From (9) and (11) we obtain the following equation for the function $\varphi(\mathbf{r})$ which is assumed to be rotationally invariant in real space (i.e. $\varphi(\mathbf{r}) = \varphi(r) e^{im\theta}$ with $m = 0$):

$$-D \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right) \varphi + D \frac{\nu^2}{r^2} \varphi + E_c e^{-U_0 \varphi^2} \varphi = E \varphi, \quad (12)$$

where $\nu = |\phi|/2\pi$. Let us introduce the dimensionless variables:

$$r = \xi x, \quad \varphi(x) = U_0^{-1/2} f(x).$$

Here $\xi^2 = D/E$ is a characteristic scale of the problem, which is nothing but the typical length of diffusion in time $t = E^{-1}$. Eq. (12) can then be written as

$$-\frac{1}{x} \frac{d}{dx} \left(x \frac{d}{dx} \right) f + \frac{\nu^2}{x^2} f + \alpha^2 e^{-f^2} f = f, \quad (13)$$

where $\alpha^2 = E_c/E$.

Since Eq. (13) is a non-linear differential equation, we are able to find only an approximate solution. As shown in Appendix, at $E \ll E_c$ (i.e. $\alpha \gg 1$) one can replace the “potential” $V(f) = \alpha^2 e^{-f^2}$ in Eq. (13) by the potential well with infinitely high walls, having the shape of a coaxial ring with the inner and outer radii x_1 and x_2 respectively. Then the instanton solution inside the ring satisfies the Schrödinger equation for a particle in the solenoid field, so that

$$f(x) = A_1 J_\nu(x) + A_2 Y_\nu(x), \quad \text{at } x_1 < x < x_2, \quad (14)$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind respectively. The positions of the matching points $x_1(\nu)$ and $x_2(\nu)$ are to be determined from the following equations:

$$\begin{cases} x_1 F_\nu(x_1) J_\nu(x_1) = -x_2 J_\nu(x_2), \\ x_1 F_\nu(x_1) Y_\nu(x_1) = -x_2 Y_\nu(x_2), \end{cases} \quad (15)$$

where

$$F_\nu(x_1) = \frac{I'_\nu(\alpha x_1)}{I_\nu(\alpha x_1)} = \frac{\nu}{\alpha x_1} + \frac{I_{\nu+1}(\alpha x_1)}{I_\nu(\alpha x_1)}$$

(the details of derivation can be found in Appendix). These equations are valid for $E \ll E_c$ and arbitrary ν . Going back to the dimensional variables, it is easy to convince oneself that the instanton action coincides with the area of the ring:

$$S_{\text{inst}} = 2\pi\rho \int_{r_1}^{r_2} dr \, r = \pi\rho\xi^2(x_2^2(\nu) - x_1^2(\nu)). \quad (16)$$

In the absence of solenoid (i.e. at $\nu = 0$), the solution of Eqs. (15) is $x_1 = 0$, $x_2 = a$, where $a \approx 2.405$ is the first zero of the function $J_0(x)$. After substitution in (16), the Lifshitz result [10] is recovered.

At $\nu \neq 0$, due to fast oscillations of the phase factor $e^{-i\phi n}$ in (6), the main contribution to the integral comes from small ϕ , which allows one to use a perturbative expansion in powers of ν . We seek a solution of (15) in the form

$$x_1 = \delta x_1(\nu), \quad x_2 = a + \delta x_2(\nu) \quad (\alpha \delta x_1 \rightarrow 0). \quad (17)$$

The Bessel functions can be expanded in powers of their index [14]:

$$J_\nu(x) = J_0(x) + \frac{\pi\nu}{2} Y_0(x) + O(\nu^2), \quad Y_\nu(x) = Y_0(x) - \frac{\pi\nu}{2} J_0(x) + O(\nu^2).$$

However, one should be careful in dealing with such expressions, since the Bessel functions are not analytical at $x = 0$, so that we are able to safely expand only the right-hand sides of Eqs. (15). Using the small- x expansions of the Bessel functions on the left-hand sides, we obtain, in the leading order in $\delta x_{1,2}$ and ν :

$$\begin{cases} \frac{1}{\alpha} \frac{1}{\Gamma(\nu)} \left(\frac{\alpha \delta x_1}{2} \right)^\nu = a J_1(a) \delta x_2 - \frac{\pi a}{2} Y_0(a) \nu, \\ \frac{1}{\alpha} \frac{\Gamma(\nu+1)}{\pi} \left(\frac{\alpha \delta x_1}{2} \right)^{-\nu} = a Y_0(a), \end{cases}$$

where $\Gamma(x)$ is the Gamma function. Therefore, the solution of Eqs. (15) looks as follows:

$$\begin{cases} x_1 = \frac{2}{\alpha} \left(\frac{\pi a Y_0(a) \alpha}{\Gamma(\nu+1)} \right)^{-1/\nu} \simeq \frac{2}{\alpha} (\pi a Y_0(a) \alpha)^{-1/\nu}, \\ x_2 = a + \left(\frac{\pi Y_0(a)}{2 J_1(a)} + \frac{1}{\pi a^2 J_1(a) Y_0(a) \alpha^2} \right) \nu \simeq a + \frac{\pi Y_0(a)}{2 J_1(a)} \nu. \end{cases} \quad (18)$$

At $\nu \rightarrow 0$ x_1 vanishes faster than δx_2 , so that its contribution to the instanton action can be neglected. From (16), we then obtain at $\phi \rightarrow 0$:

$$S_{\text{inst}}(E, \phi) = \frac{\pi \rho D a^2}{E} (1 + b|\phi| + O(\phi^2)), \quad (19)$$

where $b = \frac{Y_0(a)}{2aJ_1(a)} \approx 0.204$. A non-analytical dependence on the magnetic flux is related to the fact that one can not regard the solenoid field as a small perturbation due to r^{-2} -singularity at small r .

Finally, we see from (10) that the asymptotics of the average density of states in the presence of solenoid is given by

$$N(E, \phi) \sim \exp\left(-\frac{\pi \rho D a^2 (1 + b|\phi|)}{E}\right). \quad (20)$$

To calculate the pre-exponential factor, one should make an expansion around the instanton configuration and integrate over all non-zero modes. We, however, shall not proceed in this way further and restrict ourselves by the exponential accuracy. After substitution of (20) in (6), we arrive at

$$\mathcal{P}(n, t) \sim \int_{-\infty}^{\infty} d\phi \int_0^{\infty} dE e^{-i\phi n} e^{-Et} \exp\left(-\frac{\pi \rho D a^2 (1 + b|\phi|)}{E}\right).$$

At large t the integral over E can be calculated by the method of steepest descent, resulting in

$$\begin{aligned} \mathcal{P}(n, t) &\sim \int_{-\infty}^{\infty} d\phi e^{-i\phi n} \exp\left(-2a\sqrt{\pi \rho D t} \sqrt{1 + b|\phi|}\right) \\ &\sim \exp(-2a\sqrt{\pi \rho D t}) \left(1 + \frac{n^2}{c \rho D t}\right)^{-1}, \end{aligned} \quad (21)$$

where $c = \pi a^2 b^2 \approx 0.756$. In calculating the last integral we used the fact that the main contribution comes from small values of the flux, which justifies an expansion in powers of ϕ . The exponential factor on the right-hand side of (21) represents the asymptotic probability for a particle without solenoid to survive after a time t and coincides with the result of Balagurov and Vaks [8]. The second factor can thus be interpreted as the conditional probability for a particle which has survived to wind n times around the origin.

It is expedient to compare our results with what is known for other similar systems. For an ideal random walk without traps, the scaling variable is $x = n/\ln t$, whose asymptotic distribution is given by Spitzer's law [1]. For a self-avoiding random walk without traps, the scaling variable $x = n/\sqrt{\ln t}$ has a Gaussian distribution [15,16]. In that case, due to the hard-core repulsion, the trajectory wanders farther away from the origin than does an ideal random walk, which reduces the winding number. In our case, we see that the scaling variable is $x = n/\sqrt{t}$, and the asymptotic distribution obeys a Cauchy law. The increase of the winding number can be qualitatively understood as follows. We are considering the conditional probability, which implies that the particle has survived until the time t . This, in turn, means that it spent much of its life in a finite region of the plane almost free of traps and thus has never wandered too far away from the starting point. Such a restriction obviously results in increasing entanglement.

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APPENDIX:

We replace the "potential" $V(f) = \alpha^2 e^{-f^2}$ in the nonlinear equation (13) by a piecewise constant effective potential:

$$V_{\text{eff}}(f) = \begin{cases} \alpha^2 & , \text{ at } f < 1, \\ 0 & , \text{ at } f > 1. \end{cases} \quad (A1)$$

Then the solution is a piecewise continuous function:

$$f(x) = \begin{cases} f_1(x) & , \text{ at } 0 < x < x_1, \\ f_2(x) & , \text{ at } x_1 < x < x_2, \\ f_3(x) & , \text{ at } x_2 < x, \end{cases} \quad (A2)$$

where the functions $f_i(x)$ obey the following linear equations:

$$\begin{cases} -\frac{1}{x} \frac{d}{dx} \left(x \frac{d}{dx} \right) f_{1,3} + \frac{\nu^2}{x^2} f_{1,3} + \alpha^2 f_{1,3} = f_{1,3}, \\ -\frac{1}{x} \frac{d}{dx} \left(x \frac{d}{dx} \right) f_2 + \frac{\nu^2}{x^2} f_2 = f_2. \end{cases} \quad (\text{A3})$$

The solution $f(x)$ and its derivatives must be continuous functions of x at $x = x_{1,2}$. The positions of the matching points x_1 and x_2 are determined from the conditions $f_1(x_1) = f_2(x_1) = 1$ and $f_2(x_2) = f_3(x_2) = 1$ (see Fig. 1).

The solution of Eqs. (A3) looks as follows:

$$\begin{cases} f_1 = C_1 I_\nu(\alpha x), \\ f_2 = C_2^{(1)} J_\nu(x) + C_2^{(2)} Y_\nu(x), \\ f_3 = C_3 K_\nu(\alpha x), \end{cases} \quad (\text{A4})$$

where $I_\nu(x)$ and $K_\nu(x)$ are the Bessel functions of imaginary argument. The boundary conditions read

$$\begin{cases} f_1(x_1) = f_2(x_1) = 1, & f_2(x_2) = f_3(x_2) = 1, \\ f_1'(x_1) = f_2'(x_1), & f_2'(x_2) = f_3'(x_2). \end{cases} \quad (\text{A5})$$

Substituting (A4) in (A5), we obtain a system of six transcendent equations to determine C_1 , $C_2^{(1,2)}$, C_3 , x_1 and x_2 . After changing notations $C_2^{(1,2)} = \alpha A_{1,2}$, the equations for $A_{1,2}$ and $x_{1,2}$ take the form

$$\begin{cases} A_1 J_\nu(x_1) + A_2 Y_\nu(x_1) = \alpha^{-1}, \\ A_1 J_\nu(x_2) + A_2 Y_\nu(x_2) = \alpha^{-1}, \\ A_1 J'_\nu(x_1) + A_2 Y'_\nu(x_1) = \frac{I'_\nu(\alpha x_1)}{I_\nu(\alpha x_1)}, \\ A_1 J'_\nu(x_2) + A_2 Y'_\nu(x_2) = \frac{K'_\nu(\alpha x_2)}{K_\nu(\alpha x_2)}. \end{cases} \quad (\text{A6})$$

In the limit $\alpha \gg 1$ the right-hand sides of the first two equations vanish. If one assumes that $x_2 \sim 1$, then in the same limit the right-hand side of the last equation tends to -1 . Excluding $A_{1,2}$ from (A6), we arrive at Eqs. (15).

It is also worth explaining why we choose to split the plane into three different regions. If there were no solenoid ($\nu = 0$), then $x_1 = 0$ and the instanton solution would be given by the Bessel function $J_0(x)$ which tends to a constant at $x \rightarrow 0$. However, if $\nu \neq 0$, then the naive assumption that $x_1 = 0$ and $f(x) \sim J_\nu(x)$ is not consistent with the condition that $f(x) > 1$ everywhere inside the potential well, since $J_\nu(x) \sim x^\nu$ at $x \rightarrow 0$. For this reason one has to introduce the inner matching point $x_1 \neq 0$.

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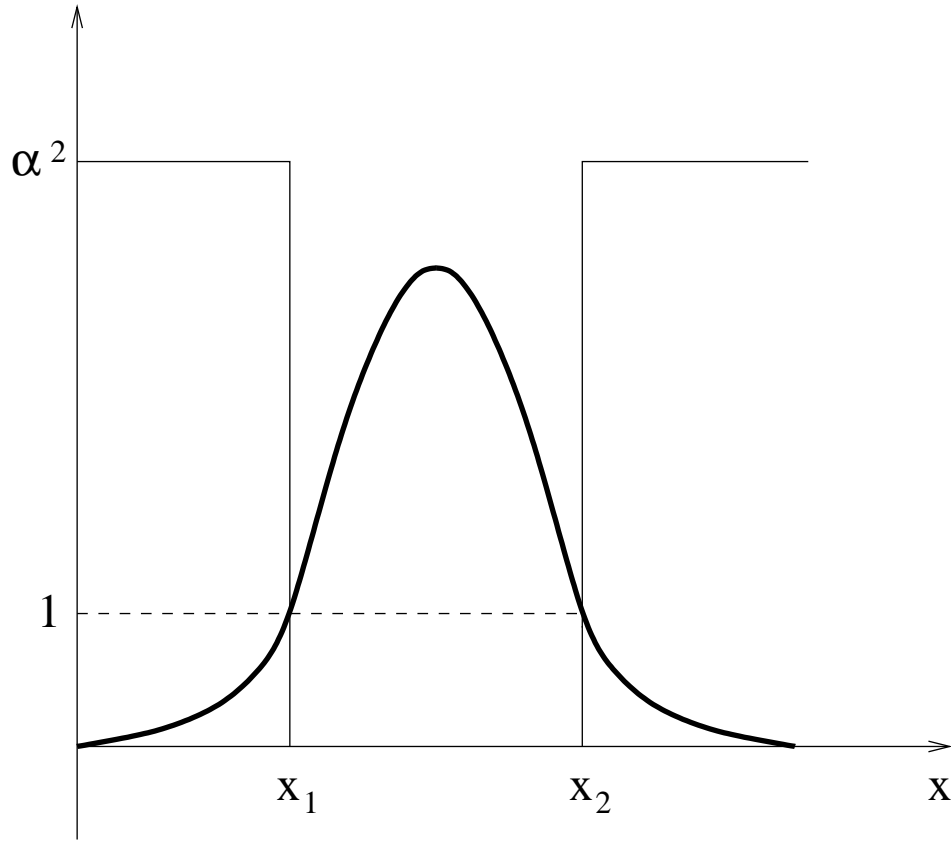


FIG. 1. The effective potential $V_{\text{eff}}(f(x))$ (light line) and the instanton solution $f(x)$ (heavy curve) as functions of $x = r/\xi$ ($\alpha^2 = E_c/E \gg 1$).